Critical transport anomalies in 4-ϵ and 6-ϵ dimensions

1975 J. Phys. A: Math. Gen. 8 L9

(http://iopscience.iop.org/0305-4470/8/1/003)

J. Phys. A: Math. Gen., Vol. 8, No. 1, 1975. Printed in Great Britain. © 1975

LETTER TO THE EDITOR

Critical transport anomalies in $4-\epsilon$ and $6-\epsilon$ dimensions

J D Gunton† and Kyozi Kawasaki

Research Institute for Fundamental Physics, Kyoto University, Kyoto 606, Japan

Received 28 November 1974

Abstract. The mode coupling equations for time-dependent correlation functions are solved to lowest order in $\epsilon = d_c - d$ for several models of critical dynamics where the 'conventional' theory is expected for the dimensionality d greater than d_c. The results are in agreement with recent renormalization group calculations. New results are obtained for the tricritical behaviour of ^3He–^4He, as well as for the critical amplitude of various transport coefficients.

Recently considerable attention has been given to extending the renormalization group approach of Wilson (1971) to dynamical critical phenomena. In particular, dynamical critical exponents have been obtained to lowest order in ϵ (where $\epsilon = d_c - d$ and d_c is the dimension such that for $d > d_c$ the conventional theory holds) for several models which possess diverging transport coefficients (Halperin *et al* 1974, Ma and Mazenko 1974), where a disagreement with earlier mode coupling predictions for the case of binary liquids (Kawasaki 1970) was found. In this note we report an alternative calculational scheme, based on the mode coupling theory, which yields the results of the renormalization group calculations in a simple manner and also gives approximate values to lowest order in ϵ for the amplitudes of the transport coefficients at the critical point. New results are also obtained for the dynamical tricritical behaviour of ^3He–^4He to lowest order in ϵ. In general our results agree with the previous predictions of mode coupling theory, with the exception of the binary liquid. In this case a certain 'constant of the motion' plays an important role, as is discussed below.

Our starting point is the coupled set of nonlinear integral equations obtained via mode coupling theory (Kawasaki 1970, 1974) for the frequency-dependent correlation functions $\{G_j(\omega)\}$ of the dynamical gross variables $\{a_j\}$. These correlation functions can be written as $G_j(\omega) = (-i\omega + i\omega_j + \gamma_j - \Sigma_j(\omega))^{-1}$ with the 'self-energy' given by

$$\Sigma_j(\omega) = -2\int_0^\infty dt \sum_{lm} \frac{\chi_l \chi_m}{\chi_j} |\mathscr{V}_{jlm}|^2 G_l(t) G_m(t)\, e^{i\omega t}. \qquad (1)$$

The transport coefficients are obtained from $\Gamma_j(\omega)$ where $\Gamma_j(\omega) = \gamma_j - \mathrm{Re}\,\Sigma_j(\omega)$. In the above the index j denotes both the wavenumber (q) dependence as well as the type of gross variable. The $\mathscr{V}_{jlm}$ are the mode coupling coefficients, $\chi_j = \langle a_j a_j^\dagger \rangle$ and the γ_j are proportional to the so-called 'bare' Onsager kinetic coefficients, L_j^0. As equation (1) is a set of complicated nonlinear integral equations, little progress has been made in solving them. We present here, however, a solution of these equations for several models of critical dynamics, in the small q, $\omega = 0$ limit, to lowest order in ϵ. For simplicity

† Supported by an NSF Visiting Scientist Fellowship. Permanent address, c/o Department of Physics, Temple University, Philadelphia, Pa 19122.

we restrict ourselves at present to $T = T_c$. Our calculation is based on the following set of approximations. First we make the Markoffian approximation,

$$G_l(t) = \exp[-(\Gamma_l + i\omega_l)t],$$

where $\Gamma_l = \Gamma_l(\omega = 0)$, so that (1) and the definition of Γ_l yield†

$$\Gamma_j = \gamma_j + 2\sum_{lm} \frac{\chi_l \chi_m}{\chi_j} |\mathscr{V}_{jlm}|^2 \frac{1}{\Gamma_l + \Gamma_m}. \tag{2}$$

Our second approximation is based on the fact that for small ϵ the wavenumbers much greater than q give the major contribution to the second term of (2), where q is the wavenumber associated with the mode j. We thus take the $q \to 0$ limit of the summand and introduce a lower cut-off in the sum over wavenumber which is proportional to q. Third, we convert the wavenumber sum to an integral and perform the angular integration (which is of the form $\int_0^\pi \sin^{d-2}\theta f(\theta)\,d\theta$) to lowest order in ϵ, with $d = d_c$. Having made these approximations we then take $\partial\Gamma_j/\partial q$, which converts the integral equations for Γ_j into simple partial differential equations which are easily solved for $\Gamma_j(q)$.

To illustrate our method we consider the simplest of our systems, the isotropic Heisenberg ferromagnet for which at $T = T_c$ there is only one independent G_j, namely $G_q(t) \equiv \langle s_{\boldsymbol{q}}^x(t) s_{-\boldsymbol{q}}^x(0)\rangle/\chi_q$, $(t > 0)$. In this case equation (2) becomes

$$L(q) = L^0 + \frac{(k_B T v_0)^2}{q^4} K_{d-1} \int_0^\Lambda dk \int_0^\pi \frac{d\theta}{2\pi} \sin^{d-2}\theta k^{d-1} \frac{\chi_k \chi_{\boldsymbol{k}-\boldsymbol{q}} (\chi_k^{-1} - \chi_{\boldsymbol{q}-\boldsymbol{k}}^{-1})^2}{\chi_q [k^4 L(k) + (\boldsymbol{q}-\boldsymbol{k})^4 L(\boldsymbol{q}-\boldsymbol{k})]}, \tag{3}$$

Λ being an upper cut-off, where we have converted the momentum integral in the usual way, with $K_{d-1} = 2^{-(d-1)}\pi^{-d/2}/\Gamma(d/2)$, and where $\Gamma(q) = q^4 L(q)$. Since $d_c = 6$, we have $\chi_k \simeq k^{-2}$ at $T = T_c$ in some dimensionless units for $\epsilon = 6 - d \ll 1$, and hence our approximations reduce (3) to the form‡

$$L(q) \simeq L^0 + g^2 \int_q^\Lambda \frac{dk}{k^{1+\epsilon} L(k)} \tag{4}$$

with $g^2 = (k_B T_c v_0)^2/192\pi^3$ where v_0 is the volume of a unit cell. Hence upon evaluating $\partial L(q)/\partial q$ of equation (4) we obtain the differential equation

$$\partial L/\partial q = -g^2/q^{1+\epsilon} L(q), \qquad q \ll 1, \tag{5}$$

whose solution is

$$L(q) = \left[L^2(q_1) + \frac{2g^2}{\epsilon}\left(\frac{1}{q^\epsilon} - \frac{1}{q_1^\epsilon}\right)\right]^{1/2}, \tag{6}$$

where q_1 is an upper limit of integration with $q < q_1 \ll 1$. Thus in the limit $q \to 0$ we find $\Gamma(q) \sim (2g^2/\epsilon)^{1/2} q^{4-\epsilon/2}$ so that the dynamical critical exponent is $z = 4 - \frac{1}{2}\epsilon = \frac{1}{2}(d+2)$, which agrees with recent renormalization group calculations§ (Ma and Mazenko 1974).

† We have neglected any dependence on ω_l and ω_m in equation (2), as they can easily be shown to be negligible in the cases considered.

‡ The lower cut-off q in the integral is arbitrary to some numerical factor, say a, of the order of unity. This results in a correction factor of $a^{-\epsilon/4}$ for $L(q)$ which can be safely ignored for small ϵ.

§ For $\epsilon = 0$, $L(q)$ still contains a logarithmic singularity, namely $L(q) \simeq 2^{1/2} g \ln q^{-1}$.

Table 1. Summary of results for the small q, $T = T_c$ behaviour of $\Gamma_j = A_j q^{z_j}$. For the Heisenberg model $\epsilon = 6-d$; otherwise $\epsilon = 4-d$. The notation used for the first four models above is similar to that of Kawasaki (1970) while that of the last model is that of Kawasaki and Gunton (1972). $\chi_{\parallel}$ is the parallel susceptibility per unit volume.

Model	Gross variables	γ_j	Dynamical critical exponent	Critical amplitude
Heisenberg	$a_1 = s_{\boldsymbol{q}}^x$	$\gamma_1 = q^2 L^0/\chi_{1q}$	$z_1 = 4 - \frac{\epsilon}{2} = \frac{d+2}{2}$	$A_1 = (2g^2/\epsilon)^{1/2}$
				$g^2 = (k_B T v_0)^2/192\pi^3$
Planar ferromagnet	$a_2 = s_{\boldsymbol{q}}^0$	$\gamma_2 = q^2 L^0$	$z_1 = z_2 = 2 - \frac{\epsilon}{2} = \frac{d}{2}$	$A_1 = (2g^2/\epsilon)^{1/2}$
	$a_1 = s_{\boldsymbol{q}}^{\dagger}$	$\gamma_1 = L^{\perp}/\chi_{1\boldsymbol{q}}$		$A_2 = A_1$
				$g^2 = (k_B T)^2/16\pi^2\chi_{\parallel}$
Superfluid ^{4}He	$a_1 = \delta\psi_{\boldsymbol{q}}^{\dagger}$	$\gamma_1 = L^0/\chi_{1\boldsymbol{q}}$	$z_1 = z_2 = 2 - \frac{\epsilon}{2} + \frac{\alpha}{2\nu}$	$A_1 = \left(\frac{10g^2}{3\epsilon}\right)^{1/2}$
	$a_2 = \delta s_{\boldsymbol{q}}$	$\gamma_2 = q^2\lambda/\rho C_p(q)$	$= \frac{1}{2}\left(d + \frac{\alpha}{\nu}\right)$	$A_2 = A_1/2$
			$(\alpha/\nu \simeq \epsilon/5)$	$g^2 = \frac{k_B(sT)^2}{16\pi^2\rho}$
Binary liquid	$a_1 = \delta v_{\boldsymbol{q}}^{\alpha}$	$\gamma_1 = \eta^0 q^2/\rho$	$z_1 = 2 - \epsilon/19$	$A_1 = \left(\frac{19k_B T}{48\pi^2\rho\epsilon}\right)^{1/19} \frac{\eta(q_1)^{18/19}}{\lambda(q_1)^{1/19}}$
	$a_2 = \delta c_{\boldsymbol{q}}$	$\gamma_2 = q^2 D^0$	$z_2 = 4 - 18\epsilon/19$	$A_2 = \left(\frac{19k_B T}{48\pi^2\rho\epsilon}\right)^{18/19} \frac{\lambda(q_1)^{1/19}}{\eta(q_1)^{18/19}}$
^{3}He–^{4}He tricritical	$a_1 = \delta\psi_{\boldsymbol{q}}^{\dagger}$	$\gamma_1 = L_{\psi}^0/\chi_{1\boldsymbol{q}}$	$z_1 = z_2 = 2 - \frac{\epsilon}{2} = \frac{d}{2}$	$A_1 = A_2 = (2g^2/\epsilon)^{1/2}$
	$a_2 = \delta x_{\boldsymbol{q}}$	$\gamma_2 = Dq^2$	$z_3 = 2 - \frac{\epsilon}{2} + \frac{\alpha_t}{\nu_t} = \frac{d}{2} + \frac{\alpha_t}{\nu_t}$	$A_3 = \frac{1}{16\pi^2}\left(\frac{m_4 x k_B T}{\rho}\right)\left(\frac{2}{\epsilon g^2}\right)^{1/2}$
	$a_3 = \delta s_{\boldsymbol{q}}$	$\gamma_3 = \kappa_s q^2/\rho C_{px}(q)$	$(\alpha_t/\nu_t = 1)$	$g^2 = \frac{1}{16\pi^2}\frac{k_B(m_4 s_4 T)}{\rho}$

We have also applied our procedure to simple models (Kawasaki 1970) which represent the critical dynamics of binary liquids, planar ferromagnets, superfluid ^{4}He and the tricritical dynamics of ^{3}He–^{4}He mixtures (Kawasaki and Gunton 1972). The calculations are completely analogous to the above except that one gets coupled differential equations. The results of our calculations are summarized in table 1. Note that in general the results are independent of Λ and hence of the 'bare' coefficients, as they should be, with one exception, namely the binary liquid. Here the amplitude of the shear viscosity $\eta(q)$ and thermal conductivity $\lambda(q)$ involves the ratio $\eta^{18/19}(q_1)/\lambda^{1/19}(q_1)$. This ratio in fact is a sort of 'constant of the motion' of the differential equations for η and λ, and is independent of q_1. In the other models considered such constants of the motion do not affect the amplitudes. This implies that in the binary liquid the mode coupling equation (or the renormalization group equation) above is not enough to determine diverging parts of transport coefficients and has to be supplemented with the initial data for $\eta^{18}(q_1)/\lambda(q_1)$. In the interesting case of three dimensions our calculational scheme loses its validity and the possibility of a diverging shear viscosity remains inconclusive. Finally we note that the results for the ^{3}He–^{4}He tricritical behaviour are new, but agree with previous mode coupling predictions (Kawasaki and Gunton 1972, Grover and Swift 1973). We hope to publish full details of our calculations later.

The authors would like to thank T Tsuneto for discussing the renormalization group approach to critical dynamics.

References

Grover M and Swift J 1973 *J. Low Temp. Phys.* **11** 751–6
Halperin B I, Hohenberg P C and Siggia E D 1974 *Phys. Rev. Lett.* **32** 1289–92
Kawasaki K 1970 *Ann. Phys.* **61** 1–56
—— 1974 *Phase Transitions and Critical Phenomena* vol 5, eds C Domb and M S Green (New York: Academic Press) to be published
Kawasaki K and Gunton J D 1972 *Phys. Rev. Lett.* **29** 1661–4
Ma S and Mazenko G F 1974 *Phys. Rev. Lett.* to be published
Wilson K G 1971 *Phys. Rev.* B **4** 3174–83